On the probability that Laplacian interface models stay positive in subcritical dimensions (Stochastic Analysis on Large Scale Interacting Systems)

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By

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Abstract

We consider a class of effective interface models on $\mathbb{Z}^d$ which is known as a model of semi-flexible membrane. The interaction depends on discrete Laplacian and the field displays huge fluctuations when $d \leq 3$. We give an estimate of the probability that the field stays positive and its behaviors differ greatly from those of the higher dimensional case or effective interface models with gradient interactions.

§ 1. Model and Result

Consider continuous spin lattice models with Laplacian interactions. Let $d \geq 1$. For a configuration $\phi = \{\phi_x\}_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, we introduce the following formal Hamiltonian:

$$H(\phi) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} V(\Delta \phi_x)^2, \tag{1.1}$$

where $V : \mathbb{R} \to \mathbb{R}$ is an interaction potential and $\Delta = \{\Delta(x,y)\}_{x,y \in \mathbb{Z}^d}$ is a discrete Laplacian on $\mathbb{Z}^d$, namely,

$$\Delta(x,y) = \begin{cases} \frac{1}{2d} & \text{if } |x-y| = 1, \\ -1 & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases}$$

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and $\Delta f(x) = \sum_{y\in\mathbb{Z}^d} \Delta(x,y)f(y)$ for $f : \mathbb{Z}^d \to \mathbb{R}$. For every $\Lambda \Subset \mathbb{Z}^d$, we define the corresponding Gibbs measure with 0-boundary conditions by

$$P_\Lambda(d\phi) = \frac{1}{Z_\Lambda}\exp\bigl\{-H(\phi)\bigr\}\prod_{x\in\Lambda} d\phi_x \prod_{x\notin\Lambda}\delta_0(d\phi_x), \tag{1.2}$$

where

$$Z_\Lambda = \int_{\mathbb{R}^\Lambda}\exp\bigl\{-H(\phi)\bigr\}\prod_{x\in\Lambda} d\phi_x \prod_{x\notin\Lambda}\delta_0(d\phi_x),$$

$d\phi_x$ denotes Lebesgue measure on $\mathbb{R}$ and δ_0 is a Dirac mass at 0. Note that though (1.1) is a formal sum, P_Λ is well-defined. In the case of $\Lambda = \Lambda_N := (-N,N)^d \cap \mathbb{Z}^d$, we denote P_Λ and Z_Λ as P_N and Z_N, respectively.

The configuration ϕ is interpreted as an effective modelization of (discretized) random interface or membrane embedded in the $d+1$-dimensional space and the spin ϕ_x denotes the height at the position $x \in \Lambda_N$. In the model of a membrane such as lipid bilayer, the energy of the interface separating the water phase and the lipid phase is given by

$$H(\phi) = \sum_{x\in\mathbb{Z}^d}\bigl\{\kappa_1(\nabla\phi_x)^2 + \kappa_2(\Delta\phi_x)^2\bigr\},$$

where κ_1 and κ_2 are called lateral tension and bending rigidity, respectively (cf. [17], [18], [20], etc.). When $\kappa_1 > 0$ and $\kappa_2 = 0$, the model is called $\nabla\phi$ model. In this case the corresponding Gibbs measure coincides with the law of the discrete Gaussian free field on $\mathbb{Z}^d$. Because of its long range correlations the field exhibits many interesting behaviors and its study has been quite active (cf. [7], [23] and references therein). On the other hand, since we consider the situation that $\kappa_1 = 0$ and $\kappa_2 > 0$, our model is called Laplacian interface model or $\Delta\phi$ model and this corresponds to (tension-less) semi-flexible membrane. Roughly speaking, the energy of interface in the $\nabla\phi$ model is determined from the surface area of the microscopic interface ϕ. $\Delta\phi$ model captures the situation that the surface area of membrane is preserved and the energy is determined from the curvature of ϕ.

In spite of physical and mathematical importance of the model, the study of $\Delta\phi$ model has been limited mainly because of the lack of several analytical tools. We first show some basic properties of $\Delta\phi$ model when the potential is quadratic. We have an equality:

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}(\Delta\phi_x)^2\Big|_{\phi\equiv 0 \text{ on } \Lambda^c} &= \sum_{x\in\overline{\Lambda}}\Bigl(\sum_{y\in\Lambda}\Delta(x,y)\phi_y\Bigr)\Bigl(\sum_{z\in\Lambda}\Delta(x,z)\phi_z\Bigr)\\ &= \sum_{y\in\Lambda}\sum_{z\in\Lambda}\Delta^2(y,z)\phi_y\phi_z\\ &= \langle\phi,\,\Delta_\Lambda^2\phi\rangle_\Lambda,\end{aligned} \tag{1.3}$$

where $\langle \cdot , \cdot \rangle_\Lambda$ denotes $l^2(\Lambda)$-inner product. We denote the outer boundary of Λ by $\partial^+\Lambda = \{x \notin \Lambda; |y-x| = 1 \text{ for some } y \in \Lambda\}$ and define $\overline{\Lambda} = \Lambda \cup \partial^+\Lambda$. Also, we define $\Delta^2 = \Delta\Delta$ as a matrix product and Δ^2_Λ denotes restriction of Δ^2 to Λ, namely, $\Delta^2_\Lambda = \{\Delta^2_\Lambda(x,y)\}_{x,y\in\mathbb{Z}^d}$ with

$$\Delta^2_\Lambda(x,y) = \begin{cases} \Delta^2(x,y) & \text{if } x,y \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, Δ_Λ denotes restriction of Δ to Λ. (1.3) shows that if the interaction potential is given by $V(r) = \frac{1}{2}r^2$, Gibbs measure P_Λ coincides with the law of a centered Gaussian field on $\mathbb{R}^\Lambda$ with covariance matrix $(\Delta^2_\Lambda)^{-1}$. The model is not a ferromagnetic spin system nor an anti-ferromagnetic because $\Delta^2(x,y)$ can be both of positive and negative for x, y with $x \neq y$. This yields that we do not have basic correlation inequalities such as FKG, Griffith, etc.. We also remark that $(\Delta^2_\Lambda)^{-1} \neq ((\Delta_\Lambda)^{-1})^2 = (-\Delta_\Lambda)^{-2}$ where we denote the product of two inverse matrices $(M^{-1})(M^{-1})$ as M^{-2} for notational convenience. The advantage of $(-\Delta_\Lambda)^{-2}$ is that we can use random walk representation. It is well-known that

$$(-\Delta_\Lambda)^{-1}(x,y) = \mathbb{E}_x\Big[\sum_{n=0}^\infty I(S_n = y, n < \tau_\Lambda)\Big],$$

where $\{S_n\}_{n\geq 0}$ is a simple random walk on $\mathbb{Z}^d$, $\mathbb{P}_x, \mathbb{E}_x$ denote its law and expectation starting at $x \in \mathbb{Z}^d$ and $\tau_\Lambda = \inf\{n \geq 0; S_n \notin \Lambda\}$ is the first exit time from $\Lambda \subset \mathbb{Z}^d$. On the other hand, random walk representation of the correlation does not hold for $(\Delta^2_\Lambda)^{-1}$ (cf. [8], [9]).

Above facts mean that basic analytical tools for spin systems or Gaussian fields do not hold for $\Delta\phi$ model and this model is much less tractable compared to $\nabla\phi$ model from the mathematical point of view. However, by comparing $(\Delta^2_N)^{-1}$ with $(\Delta_N)^{-2}$ the following estimate for the variance of the field is known (cf. [14, Section 4] and references therein). In the case of $\Lambda = \Lambda_N$, we denote Δ_{Λ_N} and $\Delta^2_{\Lambda_N}$ as Δ_N and Δ^2_N, respectively.

Proposition 1.1. *There exist constants $\gamma_d > 0$ such that*

$$\mathrm{Var}_{P_N}(\phi_0) = \begin{cases} \gamma_d + O(N^{4-d}) & \text{if } d \geq 5, \\ \gamma_d \log N + O(1) & \text{if } d = 4, \end{cases} \tag{1.4}$$

and

$$\frac{1}{\gamma_d} N^{4-d} \leq \mathrm{Var}_{P_N}(\phi_0) \leq \gamma_d N^{4-d} \quad \text{if } d \leq 3. \tag{1.5}$$

Hence the field is said to be delocalized if $d \leq 4$ because the variance diverges as $N \to \infty$ and is said to be localized if $d \geq 5$ because the variance remains finite. When $d \geq 3$ the above asymptotics corresponds to those for the $\nabla\phi$ model with dimension $d-2$. When $d \leq 3$ the model has huge fluctuations known as undulations.

Now we are in the position to state the problem and result of this paper. One of the problems related to interface is the study of behaviors of the field under the effect of various external potentials (wall, pinning, etc.). See a review [23] on the development of the study for the $\nabla\phi$ model. For $\Delta\phi$ model, by renewal type argument which is based on Markov properties of the field, detailed analysis has been made for the one-dimensional case (cf. [3], [4], [11], etc.). On the other hand, for the multi-dimensional case mathematically rigorous results are quite limited because of the lack of analytical tools as explained above. To our knowledge, only the study of entropic repulsion for $d \geq 4$ is successful and there is a partial result about pinning . In this paper below, we study the estimate on the probability that the field stays positive when $d \leq 3$. We first recall results for the higher dimensional case (cf. [12], [13], [21]). Let $V(r) = \frac{1}{2}r^2$. For every $\delta \in (0,1)$ it holds that

$$\log P_N\big(\phi_x \geq 0 \text{ for every } x \in \Lambda_{\delta N}\big) \sim \begin{cases} -C(\delta,d)N^{d-4}\log N & \text{if } d \geq 5, \\ -C(\delta,d)(\log N)^2 & \text{if } d = 4, \end{cases} \tag{1.6}$$

as $N \to \infty$ for some constant $C(\delta,d) > 0$. The corresponding result for the non-Gaussian case was also studied by [14]. When $d \leq 3$, we have the following estimate. By large fluctuation of the field, its behaviors differ greatly from those of the higher dimensional case.

Theorem 1.2. *Let $V(r) = \frac{1}{2}r^2$ and $d = 1,2,3$. For every $0 < \delta < 1$, there exist $\gamma > 0$ (small) and $C = C(\delta,\gamma,d) > 0$ such that for every $x \in \Lambda_{\delta N}$, it holds that*

$$P_N\big(\phi_y \geq 0 \textit{ for every } y \in \Lambda_{\gamma N}(x)\big) \geq C,$$

for every $N \in \mathbb{N}$ where $\Lambda_{\gamma N}(x) := x + \Lambda_{\gamma N}$.

Remark. If we assume that FKG inequality holds under P_N, then by dividing $\Lambda_{\delta N}$ into $k := \big[\frac{2\delta N+1}{2\gamma N+1}\big]^d$ small boxes with side-length $2\gamma N + 1$ and using Theorem 1.2 and FKG inequality, we have

$$P_N\big(\phi_x \geq 0 \text{ for every } y \in \Lambda_{\delta N}\big) \geq C^k > 0,$$

for every $N \in \mathbb{N}$. Therefore, we can obtain the corresponding result to the higher dimensional case (1.6). However, at this moment we don't know whether FKG inequality holds for P_N or not. It is easy to see that well-known Holley's criterion for FKG inequality (cf. [9, Appendix B]) does not hold for our Hamiltonian (1.1). Also, by [19] it is known that FKG inequality holds for Gaussian measure $\mathcal{N}(m, \Sigma)$ if and only if all elements of covariance matrix Σ are non-negative. But, in our case, inverse positivity of matrix Δ_Λ^2 for $\Lambda \subset \mathbb{Z}^d$ is a non-trivial problem and actually it depends on the underlying set Λ. For example, we can easily calculate (by using numerical software if necessary) that if we consider $T = \{(-1,0),(0,0),(1,0),(0,-1),(0,-2)\} \subset \mathbb{Z}^2$ then all elements of 5×5 matrix $(\Delta_T^2)^{-1}$ are positive. On the other hands, if we add one point to T and consider $T' = T \cup \{(0,-3)\}$ then negative elements appear in 6×6 matrix $(\Delta_{T'}^2)^{-1}$. The corresponding problem can be also considered in the continuous setting. There are extensive studies about positivity of biharmonic Green's functions (cf. [10] and references therein).

For the measure $Q_N \sim \mathcal{N}(0, (-\Delta_N)^{-2})$, FKG inequality holds since all the elements of $(-\Delta_N)^{-2}$ are positive by random walk representation for $(-\Delta_N)^{-1}$. Our proof of Theorem 1.2 below also works well for this measure and we can prove that when $d \leq 3$, for every $0 < \delta < 1$ there exists $C = C(\delta, d) > 0$ such that

$$Q_N\big(\phi_x \geq 0 \text{ for every } x \in \Lambda_{\delta N}\big) \geq C,$$

for every $N \in \mathbb{N}$.

Next, for the one-dimensional case we can prove that the corresponding result holds for general interaction potentials V.

Theorem 1.3. *Let $d = 1$ and assume that $V : \mathbb{R} \to \mathbb{R}$ satisfies the condition that $r \mapsto e^{-V(r)}$ is bounded and continuous, $\int_{\mathbb{R}} e^{-V(r)} dr < \infty$, $\int_{\mathbb{R}} r e^{-V(r)} dr = 0$ and $\int_{\mathbb{R}} r^2 e^{-V(r)} dr < \infty$. Then, for every $0 < \delta < 1$ there exists $C = C(\delta) > 0$ such that*

$$P_N\big(\phi_x \geq 0 \textit{ for every } x \in \Lambda_{\delta N}\big) \geq C,$$

for every $N \in \mathbb{N}$.

Remark. It is well-known that Laplacian interface model in one dimension can be represented as integrated random walk. Let $\{X_n\}_{n \geq 1}$ be i.i.d. $\mathbb{R}$-valued random variables whose distribution is given by $P(X_1 \in dx) = \frac{1}{Z} e^{-V(x)} dx$. By the assumption on V, $E[X_1] = 0$ and $\sigma^2 := E[(X_1)^2] \in (0, \infty)$. For $a, b \in \mathbb{R}$, define $Y_n := a + \sum_{i=1}^n X_i$ and $Z_n := b + \sum_{i=1}^n Y_i = an + b + \sum_{i=1}^n (n - i + 1) X_i, n \geq 1$. $\mu^{(a,b)}$ denotes the law of the integrated random walk $\{Z_n\}$. Then, by [3, Proposition 2.2], $P_{[1,N-1]}(\,\cdot\,)$ and the law of $(Z_1, Z_2, \cdots, Z_{N-1})$ under $\mu^{(0,0)}(\,\cdot\,| Z_N = 0, Z_{N+1} = 0)$ are same. Therefore, Theorem 1.3 means that for every $0 < s < t < 1$, there exists $C = C(s,t) > 0$ such that

$$\mu\big(Z_n \geq 0 \text{ for every } sN \leq n \leq tN \;\big|\; Z_N = 0, Z_{N+1} = 0\big) \geq C,$$

for every $N \in \mathbb{N}$.

To study the asymptotics of the probability of $\{X_n \geq 0$ for every $1 \leq n \leq N\}$ as $N \to \infty$ for a real valued stochastic process $\{X_n\}_{n \geq 0}$ is called the problem of persistence probability or survival probability and this problem has been actively investigated for many stochastic processes, recently (cf. [2] and references therein). In particular, for integrated random walk $\{Z_n\}_{n \geq 0}$, it has been proved that under some assumptions on X_n,

$$\mu\big(Z_n \geq 0 \text{ for every } 1 \leq n \leq N\big) \asymp \frac{1}{N^{\frac{1}{4}}},$$

and

$$\mu\big(Z_n \geq 0 \text{ for every } 1 \leq n \leq N \bigm| Z_N = 0, Z_{N+1} = 0\big) \asymp \frac{1}{N^{\frac{1}{2}}},$$

as $N \to \infty$ (cf. [3], [5], [6], etc.).

Remark. Estimates on the probability of $\{\phi_x \geq 0$ for every $x \in \Lambda_{\delta N}\}$ closely related to phenomenon called entropic repulsion. Namely, for $\nabla\phi$ model in $d \geq 2$ and $\Delta\phi$ model in $d \geq 4$, it has been proved that the field is pushed up to the same level as the maximum of the field under this hard wall condition (cf. [7], [12], [13], [23], etc.). On the other hand, by (1.5) and Lemma 2.1 below we can see that maximum and square root of variance of the field are the same order for $\Delta\phi$ model in $d \leq 3$. Therefore entropic repulsion is not expected to occur in this case.

In the rest of this paper, we give the proof of Theorem 1.2 in Section 2 and give the proof of Theorem 1.3 in Section 3. We remark that throughout this paper below, C represents a positive constant which does not depend on the size of the system N, but may depend on other parameters. This C in estimates may change from place to place in the paper.

§ 2. Proof of Theorem 1.2

In this section we assume that $V(r) = \frac{1}{2}r^2$. We first give estimates on L^2-distance and maximum of the field.

Lemma 2.1. *Let $d = 1, 2, 3$. Take $0 < \delta < 1$, $x \in \Lambda_{\delta N}$ and $0 < \gamma < 1$ which satisfy $\Lambda_{\gamma N}(x) \subset \Lambda_N$. Then, there exist $C_1 = C_1(\delta, \gamma, d) > 0, C_2 = C_2(\delta, \gamma, d) > 0$ such that the following holds for every $N \in \mathbb{N}$.*

$$\max_{y \in \Lambda_{\gamma N}(x)} E^{P_N}\left[(\phi_y - \phi_x)^2\right] \leq C_1 (a_N)^2, \tag{2.1}$$

and

$$E^{P_N}\Big[\max_{y \in \Lambda_{\gamma N}(x)} \phi_y\Big] \leq C_2 a_N, \tag{2.2}$$

where $a_N = N^{\frac{4-d}{2}}$.

Remark. By the proof of this lemma we can see that constants $C_1, C_2 \to 0$ as $\gamma \to 0$.

The proof of this lemma is given later. Once we have Lemma 2.1, Theorem 1.2 can be proved by combination of several Gaussian techniques.

Proof of Theorem 1.2. Let $x \in \Lambda_{\delta N}$ and set $\sigma_N^2(x) := E^{P_N}\big[(\phi_x)^2\big]$. For every $\alpha > 0$, we have

$$\begin{aligned}&\big\{\phi_x \geq \alpha\sqrt{\sigma_N^2(x)}\big\}\\&\quad\subset \big\{\phi_y \geq 0 \text{ for every } y \in \Lambda_{\gamma N}(x)\big\} \cup \big\{\max_{y\in\Lambda_{\gamma N}(x)}\{\phi_x - \phi_y\} \geq \alpha\sqrt{\sigma_N^2(x)}\big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&P_N\big(\phi_y \geq 0 \text{ for every } y \in \Lambda_{\gamma N}(x)\big)\\&\quad\geq P_N\big(\phi_x \geq \alpha\sqrt{\sigma_N^2(x)}\big) - P_N\big(\max_{y\in\Lambda_{\gamma N}(x)}\{\phi_x - \phi_y\} \geq \alpha\sqrt{\sigma_N^2(x)}\big).\end{aligned}$$

By Gaussian tail estimate,

$$P_N\big(\phi_x \geq \alpha\sqrt{\sigma_N^2(x)}\big) = \int_\alpha^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\,du \geq \big(\frac{1}{\alpha} - \frac{1}{\alpha^3}\big)\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\alpha^2}.$$

Also,

$$\begin{aligned}&P_N\Big(\max_{y\in\Lambda_{\gamma N}(x)}\{\phi_x - \phi_y\} \geq \alpha\sqrt{\sigma_N^2(x)}\Big)\\&\quad= P_N\Big(\max_{y\in\Lambda_{\gamma N}(x)}\{\phi_y - \phi_x\} - E^{P_N}\big[\max_{y\in\Lambda_{\gamma N}(x)}\{\phi_y - \phi_x\}\big]\\&\qquad\qquad\geq \alpha\sqrt{\sigma_N^2(x)} - E^{P_N}\big[\max_{y\in\Lambda_{\gamma N}(x)}\{\phi_y - \phi_x\}\big]\Big)\\&\quad\leq \exp\Big\{-\frac{(\beta_N)^2}{2\max\limits_{y\in\Lambda_{\gamma N}(x)} E^{P_N}\big[(\phi_y - \phi_x)^2\big]}\Big\},\end{aligned}$$

where we set $\beta_N := \max\big\{\alpha\sqrt{\sigma_N^2(x)} - E^{P_N}\big[\max_{y\in\Lambda_{\gamma N}(x)}\{\phi_y - \phi_x\}\big], 0\big\}$. We used symmetry for the first equality and Borell's inequality (cf. [1, Theorem 2.1.1]) for the last inequality.

Now, we have $E^{P_\Gamma}[(\phi_x)^2] \leq E^{P_\Lambda}[(\phi_x)^2]$ for every $x \in \Gamma \subset \Lambda \Subset \mathbb{Z}^d$ (cf. [22, (14) and its proof]). Since $\Lambda_{(1-\delta)N}(x) \subset \Lambda_N$ for $x \in \Lambda_{\delta N}$, by this comparison estimate and (1.5),

$$E^{P_N}\big[(\phi_x)^2\big] \geq E^{P_{\Lambda_{(1-\delta)N}(x)}}\big[(\phi_x)^2\big] = E^{P_{(1-\delta)N}}\big[(\phi_0)^2\big] \geq C_3 N^{4-d},$$

for some $C_3 = C_3(\delta) > 0$. Combining this with Lemma 2.1, we have

$$\frac{(\beta_N)^2}{\max_{y \in \Lambda_{\gamma N}(x)} E^{P_N}\left[(\phi_y - \phi_x)^2\right]} \geq \frac{(\max\{\alpha\sqrt{C_3} - C_2, 0\})^2}{C_1}.$$

Since $C_1, C_2 \to 0$ as $\gamma \to 0$, we can take $\gamma > 0$ small enough so that $\frac{(\max\{\alpha\sqrt{C_3} - C_2, 0\})^2}{C_1} \gg \alpha^2$. By collecting all the estimates, we have

$$P_N\big(\phi_y \geq 0 \text{ for every } y \in \Lambda_{\gamma N}(x)\big) \geq C,$$

for some constant $C = C(\delta, \gamma, d) > 0$. □

Next, we prove Lemma 2.1. The key is the following simple comparison estimate. Recall that P_N and Q_N denote the law of centered Gaussian fields on $\mathbb{R}^{\Lambda_N}$ whose covariance matrices are given by $(\Delta_N^2)^{-1}$ and $(\Delta_N)^{-2}$, respectively.

Lemma 2.2. *For every $x, y \in \Lambda_N$, we have*

$$E^{P_N}\left[(\phi_y - \phi_x)^2\right] \leq E^{Q_N}\left[(\phi_y - \phi_x)^2\right].$$

Proof. At first, for every $\Lambda \Subset \mathbb{Z}^d$ we have

$$\begin{aligned}\sum_{x \in \mathbb{Z}^d} (\Delta\phi_x)^2\Big|_{\phi \equiv 0 \text{ on } \Lambda^c} &= \sum_{x \in \Lambda}\Big(\sum_{y \in \Lambda} \Delta(x, y)\phi_y\Big)\Big(\sum_{z \in \Lambda} \Delta(x, z)\phi_z\Big) \\ &\quad + \sum_{x \in \partial^+\Lambda}\Big(\sum_{y \in \Lambda} \Delta(x, y)\phi_y\Big)\Big(\sum_{z \in \Lambda} \Delta(x, z)\phi_z\Big) \\ &= \langle \phi, (-\Delta_\Lambda)^2\phi\rangle_\Lambda + B_\Lambda(\phi),\end{aligned}$$

where

$$B_\Lambda(\phi) = \sum_{x \in \partial^+\Lambda}\Big(\frac{1}{2d}\sum_{\substack{y \in \Lambda \\ |x-y|=1}} \phi_y\Big)^2. \tag{2.3}$$

In particular, in the case of $\Lambda = \Lambda_N$,

$$B_{\Lambda_N}(\phi) = \sum_{x \in \partial^-\Lambda_N} \frac{r_N(x)}{4d^2}(\phi_x)^2,$$

where $r_N(x) = \#\{y \in \partial^+\Lambda_N; |y - x| = 1\}$ and $\partial^-\Lambda = \{x \in \Lambda; |y - x| = 1 \text{ for some } y \notin \Lambda\}$ denotes the inner boundary of $\Lambda \subset \mathbb{Z}^d$. By comparing this with (1.3), we obtain an identity $\Delta_N^2 = (-\Delta_N)^2 + B_N$, where $B_N = (B_N(x, y))_{x,y \in \Lambda_N}$ is a $|\Lambda_N| \times |\Lambda_N|$ matrix given by

$$B_N(x, y) = \begin{cases} \frac{r_N(x)}{4d^2} & \text{if } x = y \in \partial^-\Lambda_N, \\ 0 & \text{otherwise.}\end{cases}$$

This identity yields that $(-\Delta_N)^{-2} - (\Delta_N^2)^{-1}$ is a non-negative definite matrix, i.e., $\langle \phi, (-\Delta_N)^{-2}\phi\rangle_{\Lambda_N} \geq \langle \phi, (\Delta_N^2)^{-1}\phi\rangle_{\Lambda_N}$ for every $\phi \in \mathbb{R}^{\Lambda_N}$. Because,

$$\begin{aligned}
&\langle \phi, ((-\Delta_N)^{-2} - (\Delta_N^2)^{-1})\phi\rangle_{\Lambda_N} \\
&\quad = \langle \phi, (-\Delta_N)^{-2}(\Delta_N^2 - (-\Delta_N)^2)(\Delta_N^2)^{-1}\phi\rangle_{\Lambda_N} \\
&\quad = \langle \phi, (-\Delta_N)^{-2}B_N(\Delta_N^2)^{-1}\phi\rangle_{\Lambda_N} \\
&\quad = \langle (\Delta_N^2)\psi, (-\Delta_N)^{-2}B_N\psi\rangle_{\Lambda_N} \\
&\quad = \langle (-\Delta_N)^2\psi, (-\Delta_N)^{-2}B_N\psi\rangle_{\Lambda_N} + \langle B_N\psi, (-\Delta_N)^{-2}B_N\psi\rangle_{\Lambda_N} \\
&\quad = \langle \psi, B_N\psi\rangle_{\Lambda_N} + \langle B_N\psi, (-\Delta_N)^{-2}B_N\psi\rangle_{\Lambda_N} \\
&\quad \geq 0,
\end{aligned}$$

where we set $\psi = (\Delta_N^2)^{-1}\phi$. Note that $(-\Delta_N)^{-2}, B_N$ are symmetric and non-negative definite. Since $E^{P_N}\big[\phi_x\phi_y\big] = (\Delta_N^2)^{-1}(x,y)$ and $E^{Q_N}\big[\phi_x\phi_y\big] = (-\Delta_N)^{-2}(x,y)$ for every $x, y \in \Lambda_N$, we obtain the result. $\square$

Remark. By this proof we can see that P_N is obtained by adding convex self-potentials at the boundary $\partial^-\Lambda_N$ to the Gaussian measure on $\mathbb{R}^{\Lambda_N}$ with the covariance $(-\Delta_N)^{-2}$.

By Lemma 2.2 and Sudakov-Fernique inequality (cf. [1, Theorem 2.2.3]), we have $E^{P_N}\big[\max_{x\in\Gamma}\phi_x\big] \leq E^{Q_N}\big[\max_{x\in\Gamma}\phi_x\big]$ for every $\Gamma \subset \Lambda_N$. Therefore, we have only to prove Lemma 2.1 for the measure Q_N instead of P_N. For this proof we follow the argument of [16] which showed the corresponding result under the measure with periodic boundary conditions. We recall that eigenvalues and eigenfunctions of Δ_N are explicitly known (cf. [15, Section 8.2]).

Lemma 2.3. *For $s = (s^i)_{1\leq i\leq d} \in \Lambda_N$, define*

$$f_s(x) := \frac{1}{N^{\frac{d}{2}}}\prod_{i=1}^{d}\sin\frac{(x^i+N)(s^i+N)\pi}{2N}, \quad x = (x^i)_{1\leq i\leq d} \in \Lambda_N.$$

Then, f_s is an eigenfunction of Δ_N and the corresponding eigenvalue is given by

$$\alpha_s := -\frac{2}{d}\sum_{i=1}^{d}\Big(\sin\frac{(s^i+N)\pi}{4N}\Big)^2.$$

Moreover, $\{f_s\}_{s\in\Lambda_N}$ forms an orthonormal basis of $\mathbb{R}^{\Lambda_N}$ equipped with inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\Lambda_N}$.

Now, define $|\Lambda_N| \times |\Lambda_N|$-matrices $U_N = (U_N(x,s))_{x,s\in\Lambda_N}$ by $U_N(x,s) := f_s(x)$ and $D_N = (D_N(x,s))_{x,s\in\Lambda_N}$ by $D_N(x,s) := \alpha_s\delta(x,s)$. By diagonalization of Δ_N, we have

$U_N^{-1}\Delta_N U_N = D_N$ and this yields $(\Delta_N)^{-2} = U_N D_N^{-2} U_N$. Note that U_N is a symmetric orthogonal matrix by Lemma 2.3. Therefore, we have the following representation of covariance under the Gaussian measure Q_N:

$$E^{Q_N}[\phi_x \phi_y] = \big(U_N D_N^{-2} U_N\big)(x,y) = \sum_{s\in\Lambda_N} \frac{1}{\alpha_s^2} f_s(x) f_s(y), \tag{2.4}$$

for every $x, y \in \Lambda_N$.

Lemma 2.4. *For every $x, y \in \Lambda_N$, we have*

$$E^{Q_N}[(\phi_x - \phi_y)^2] \le \frac{1}{N^d} \sum_{s\in\Lambda_N} \frac{1}{\alpha_s^2} h_{x-y}(s),$$

where $h_z(s) := \min\big\{\frac{\pi^2}{4}\|z\|^2\|\frac{s+N}{N}\|^2,\ 4\big\}$, $z, s \in \Lambda_N$. $\|\cdot\|$ denotes l^2-norm of $\mathbb{R}^{\Lambda_N}$.

Proof. By (2.4), we have

$$\begin{aligned}
&E^{Q_N}[(\phi_x - \phi_y)^2] \\
&= \sum_{s\in\Lambda_N} \frac{1}{\alpha_s^2} (f_s(x) - f_s(y))^2 \\
&= \frac{1}{N^d} \sum_{s\in\Lambda_N} \frac{1}{\alpha_s^2} \Big\{ \prod_{i=1}^d \sin\frac{(x^i+N)(s^i+N)\pi}{2N} - \prod_{i=1}^d \sin\frac{(y^i+N)(s^i+N)\pi}{2N} \Big\}^2.
\end{aligned}$$

Therefore, it is sufficient to show that $\{\cdots\}^2 \le h_{x-y}(s)$ for every $s \in \Lambda_N$. It is easy to see that for every real sequences $\{a_i\}_{1\le i\le n}, \{b_i\}_{1\le i\le n}$ which satisfy $|a_i|, |b_i| \le 1$ for every $1 \le i \le n$, we have $\big|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\big| \le \sum_{i=1}^n |a_i - b_i|$. Therefore,

$$\begin{aligned}
&\Big| \prod_{i=1}^d \sin\frac{(x^i+N)(s^i+N)\pi}{2N} - \prod_{i=1}^d \sin\frac{(y^i+N)(s^i+N)\pi}{2N} \Big| \\
&\le \sum_{i=1}^d \Big| \sin\frac{(x^i+N)(s^i+N)\pi}{2N} - \sin\frac{(y^i+N)(s^i+N)\pi}{2N} \Big| \\
&= \sum_{i=1}^d \Big| 2\cos\frac{(x^i+y^i+2N)(s^i+N)\pi}{4N} \sin\frac{(x^i-y^i)(s^i+N)\pi}{4N} \Big| \\
&\le \sum_{i=1}^d \frac{|x^i-y^i||s^i+N|\pi}{2N} \\
&\le \frac{\pi}{2}\|x-y\|\|\frac{s+N}{N}\|,
\end{aligned}$$

where the last inequality follows from Schwarz's inequality. Another bound is trivial. □

Proof of Lemma 2.1. For $x \in \mathbb{R}^d$ and $r > 0$, define $B_\infty(x, r) := \{y \in \mathbb{R}^d; \|y - x\|_\infty < r\}$. By Lemma 2.4, we have

$$E^{Q_N}[(\phi_x - \phi_y)^2] \le \int_{\mathbb{R}^d} \sum_{s \in \Lambda_N} \frac{1}{\alpha_s^2} h_{x-y}(s) 1_{B_\infty(\frac{s}{N}, \frac{1}{2N})}(u) du.$$

For every $s \in \Lambda_N$ and $1 \le i \le d$, $|\frac{(s^i+N)\pi}{4N}| < \frac{\pi}{2}$. Then, by the fact that $\frac{2}{\pi}|\theta| \le |\sin\theta|$ for every $|\theta| \le \frac{\pi}{2}$, we have

$$\frac{1}{\alpha_s^2} \le \frac{d^2}{4} \frac{1}{\left\{\sum_{i=1}^{d} \left(\frac{(s^i+N)}{2N}\right)^2\right\}^2} = \frac{d^2}{\|\frac{s+N}{N}\|^4}.$$

Also, for every $s \in \Lambda_N$ and $u \in B_\infty(\frac{s}{N}, \frac{1}{2N})$, it is easy to see that $\|u + 1\| \le 2\|\frac{s+N}{N}\|$ and $\frac{1}{2N} \le \|u + 1\| \le 2\sqrt{d}$. By combining all the estimates, we have

$$\begin{aligned}
&E^{Q_N}[(\phi_x - \phi_y)^2] \\
&\le C \int_{\frac{1}{2N} \le \|u+1\| \le 2\sqrt{d}} \sum_{s \in \Lambda_N} \min\left\{\frac{1}{\|u+1\|^4}, \frac{\|x-y\|^2}{\|u+1\|^2}\right\} 1_{B_\infty(\frac{s}{N}, \frac{1}{2N})}(u) du \\
&\le C \int_{\frac{1}{2N} \le \|u+1\| \le \beta_N} \frac{\|x-y\|^2}{\|u+1\|^2} du + C \int_{\beta_N \le \|u+1\| \le 2\sqrt{d}} \frac{1}{\|u+1\|^4} du \\
&=: I_N^1 + I_N^2,
\end{aligned}$$

for some $C = C(d) > 0$, where β_N satisfies $\frac{1}{2N} \le \beta_N \le 2\sqrt{d}$ and is specified later on. Each integral is estimated as

$$I_N^1 \le C\|x-y\|^2 \int_{\frac{1}{2N}}^{\beta_N} \frac{1}{r^2} r^{d-1} dr \le \begin{cases} C\|x-y\|^2 N & \text{if } d = 1, \\ C\|x-y\|^2 \log(C\beta_N N) & \text{if } d = 2, \\ C\|x-y\|^2 \beta_N^{d-2} & \text{if } d \ge 3, \end{cases}$$

and

$$I_N^2 \le C \int_{\beta_N}^{2\sqrt{d}} \frac{1}{r^4} r^{d-1} dr \le \begin{cases} C\beta_N^{d-4} & \text{if } d \le 3, \\ C\log(\frac{C}{\beta_N}) & \text{if } d = 4, \\ C & \text{if } d \ge 5. \end{cases}$$

Now we choose β_N as $\beta_N = \frac{\sqrt{d}}{\|x-y\|}$. Since $1 \le \|x - y\| \le \sqrt{d}\gamma N$ for every $x \in \Lambda_{\delta N}$ and $y \in \Lambda_{\gamma N}(x)$ $(x \ne y)$, this choice of β_N satisfies the condition $\frac{1}{2N} \le \beta_N \le 2\sqrt{d}$ and by these estimates we obtain (2.1).

By using Dudley's bound the proof of (2.2) follows from the completely same way as [16, Proposition 8.3] with above estimates also. □

§ 3. Proof of Theorem 1.3

In this section, we assume that $d = 1$ and potential $V : \mathbb{R} \to \mathbb{R}$ satisfies the condition that $r \mapsto e^{-V(r)}$ is bounded and continuous, $Z := \int_{\mathbb{R}} e^{-V(r)} dr < \infty$, $\int_{\mathbb{R}} re^{-V(r)} dr = 0$ and $\int_{\mathbb{R}} r^2 e^{-V(r)} dr < \infty$. For notational simplicity we consider the model on $\widetilde{\Lambda}_N := [1, N-1] \cap \mathbb{Z}$ with 0-boundary conditions. Define

$$\mathbb{P}_N(d\phi|a,b,c,d) := \frac{1}{Z_N(a,b,c,d)} \exp\bigl\{-\mathcal{H}_{[-1,N+1]}(\phi)\bigr\} \times \prod_{x=1}^{N-1} d\phi_x \delta_a(d\phi_{-1})\delta_b(d\phi_0)\delta_c(d\phi_N)\delta_d(d\phi_{N+1}),$$

where $\mathcal{H}_{[l,r]}(\phi) := \sum_{x=l+1}^{r-1} V(\Delta\phi_x)$ and $Z_N(a,b,c,d)$ is a normalization factor. Then the Gibbs measure (1.2) in the one-dimensional case is given by $P_{\widetilde{\Lambda}_N}(d\phi) = \mathbb{P}_N(d\phi|0,0,0,0)$. We denote $\mathbb{P}_N(d\phi) := \mathbb{P}_N(d\phi|0,0,0,0)$ for simplicity. We also introduce i.i.d. $\mathbb{R}$-valued random variables $\{X_n\}_{n\geq 1}$ whose distribution is given by $P(X_1 \in dx) = \frac{1}{Z} e^{-V(x)} dx$. By the assumption on V, $E[X_1] = 0$ and $\sigma^2 := E[(X_1)^2] \in (0,\infty)$. Theorem 1.3 follows from the functional central limit theorem studied by [11].

Proof of Theorem 1.3. By change of variables we may assume that $\sigma^2 = 1$. Let $0 < s < t < 1$. For $\alpha > 0, \beta > 0$, define an event

$$\mathcal{A}_N := \bigl\{\alpha N^{\frac{3}{2}} \leq \phi_{[sN]} \leq (\alpha+\beta) N^{\frac{3}{2}},\ \alpha N^{\frac{3}{2}} \leq \phi_{[tN]} \leq (\alpha+\beta) N^{\frac{3}{2}}, \\ |\phi_{[sN]} - \phi_{[sN]-1}| \leq \beta N^{\frac{1}{2}},\ |\phi_{[tN]} - \phi_{[tN]-1}| \leq \beta N^{\frac{1}{2}}\bigr\}.$$

Then, we have

$$\begin{aligned}
&\mathbb{P}_N\bigl(\phi_x \geq 0 \text{ for every } [sN] \leq x \leq [tN]\bigr) \\
&\quad \geq \mathbb{P}_N\bigl(\phi_x \geq 0 \text{ for every } [sN] \leq x \leq [tN] \bigm| \mathcal{A}_N\bigr) \cdot \mathbb{P}_N\bigl(\mathcal{A}_N\bigr) \\
&\quad =: I_N^1 \cdot I_N^2.
\end{aligned}$$

We show that for appropriate choice of α and β, I_N^1 and I_N^2 are positive uniformly in N.

Estimate on I_N^1. Set $r := \frac{1}{t-s} > 1$. By Markov property of the model with distance 2 and replacing $(t-s)N$ by N, for the estimate on I_N^1, it is sufficient to prove that

$$\begin{aligned}
\widetilde{I}_N^1 &:= \mathbb{P}_N\bigl(\phi_x \geq 0 \text{ for every } 1 \leq x \leq N-1 \bigm| \\
&\qquad ar^{\frac{3}{2}} N^{\frac{3}{2}} - \xi^L r^{\frac{1}{2}} N^{\frac{1}{2}},\ ar^{\frac{3}{2}} N^{\frac{3}{2}},\ br^{\frac{3}{2}} N^{\frac{3}{2}} - \xi^R r^{\frac{1}{2}} N^{\frac{1}{2}},\ br^{\frac{3}{2}} N^{\frac{3}{2}}\bigr) \\
&\geq C > 0,
\end{aligned}$$

for every $N \in \mathbb{N}$ uniformly in a, b, ξ^L, ξ^R which satisfy

$$a \in [\alpha, \alpha+\beta],\ b \in [\alpha, \alpha+\beta],\ \xi^L \in [-\beta, \beta],\ \xi^R \in [-\beta, \beta]. \tag{3.1}$$

By translation invariance in height variables, we have

$$\begin{aligned}\widetilde{I}_N^1 = \mathbb{P}_N\big(\phi_x \ge -ar^{\frac{3}{2}}N^{\frac{3}{2}} + \xi^L r^{\frac{1}{2}}N^{\frac{1}{2}} \text{ for every } 1 \le x \le N-1 \bigm| 0,\ \xi^L r^{\frac{1}{2}}N^{\frac{1}{2}},\\ (b-a)r^{\frac{3}{2}}N^{\frac{3}{2}} - (\xi^R - \xi^L)r^{\frac{1}{2}}N^{\frac{1}{2}},\ (b-a)r^{\frac{3}{2}}N^{\frac{3}{2}} + \xi^L r^{\frac{1}{2}}N^{\frac{1}{2}}\big).\end{aligned} \tag{3.2}$$

We recall the functional central limit theorem [11, Corollary 2.2] which is applied to our setting

Theorem 3.1. *Let $\{\xi_N^L\}_{N\ge1}, \{\xi_N^R\}_{N\ge1}, \{d_N\}_{N\ge1}$ be sequences of real number which satisfy $-\frac{\xi_N^L+\xi_N^R}{\sqrt{N}} \to u \in \mathbb{R}$ and $\frac{1}{\sqrt{N}}(\frac{d_N}{N} - \xi_N^L) \to v \in \mathbb{R}$ as $N \to \infty$. Then the law of $\{\frac{1}{N^{\frac{3}{2}}}(\phi_{[Nt]} - [Nt]\xi_N^L)\}_{t\in[0,1]}$ under the measure $\mathbb{P}_N(\,\cdot\,|0, \xi_N^L, d_N + \xi_N^R, d_N)$ converges weakly in $C([0,1];\mathbb{R})$ to a Gaussian process $\{\theta(t)\}_{t\in[0,1]}$ whose mean and covariance are given by*

$$\begin{aligned} m(t) &:= E[\theta(t)] = t^2(t-1)u + t^3(3-2t)v,\\ g(s,t) &:= \mathrm{Cov}(\theta(s), \theta(t)) = \tfrac{s^2(1-t)^2}{6}\{2t(1-s) + t - s\},\end{aligned}$$

for $0 \le s \le t \le 1$.

By using this theorem for (3.2), we have

$$\begin{aligned}\liminf_{N\to\infty} \widetilde{I}_N^1 &\ge P\big(\theta(t) \ge -\xi^L r^{\frac{1}{2}} t - ar^{\frac{3}{2}} \text{ for every } 0 \le t \le 1\big)\\ &\ge P\big(\theta(t) - m(t) \ge -r^{\frac{3}{2}}(\alpha - 9\beta) \text{ for every } 0 \le t \le 1\big)\\ &=: q,\end{aligned}$$

where $\{\theta(t)\}_{t\in[0,1]}$ is a Gaussian process of Theorem 3.1 with $u = r^{\frac{1}{2}}(\xi^R - \xi^L), v = r^{\frac{3}{2}}(b-a) - r^{\frac{1}{2}}\xi^L$. The last inequality follows from $\inf_{t\in[0,1]}\big\{\xi^L r^{\frac{1}{2}} t + ar^{\frac{3}{2}} + m(t)\big\} \ge r^{\frac{3}{2}}(\alpha - 9\beta)$ under the condition (3.1). Now, $\{\widetilde{\theta}(t)\}_{t\in[0,1]} := \{\theta(t) - m(t)\}_{t\in[0,1]}$ is a centered Gaussian process with bounded covariance $\{g(s,t)\}_{s,t\in[0,1]}$. Therefore Dudley's bound yields that $E\big[\sup_{t\in[0,1]} \widetilde{\theta}(t)\big] \le C_0$ for some $C_0 > 0$. If we choose $\alpha > 0$ large enough and $\beta > 0$ small enough so that $r^{\frac{3}{2}}(\alpha - 9\beta) \ge 2C_0$, then by symmetry and Borell's inequality we have

$$\begin{aligned} q &= 1 - P\Big(\sup_{t\in[0,1]} \widetilde{\theta}(t) - E\big[\sup_{t\in[0,1]} \widetilde{\theta}(t)\big] \ge r^{\frac{3}{2}}(\alpha - 9\beta) - E\big[\sup_{t\in[0,1]} \widetilde{\theta}(t)\big]\Big)\\ &\ge 1 - P\Big(\sup_{t\in[0,1]} \widetilde{\theta}(t) - E\big[\sup_{t\in[0,1]} \widetilde{\theta}(t)\big] \ge C_0\Big)\\ &\ge C_1,\end{aligned}$$

for some $C_1 > 0$ uniformly in a, b, ξ^L, ξ^R which satisfy the condition (3.1).

Estimate on I_N^2. By the relation between one-dimensional Laplacian interface model and integrated random walk, we have

$$
\begin{aligned}
I_N^2 = \mu^{(0,0)}\Big(&\alpha N^{\frac{3}{2}} \le Z_{[sN]} \le (\alpha+\beta)N^{\frac{3}{2}}, \alpha N^{\frac{3}{2}} \le Z_{[tN]} \le (\alpha+\beta)N^{\frac{3}{2}}, \\
&|Y_{[sN]}| \le \beta N^{\frac{1}{2}}, |Y_{[tN]}| \le \beta N^{\frac{1}{2}} \;\Big|\; Z_N = 0, Z_{N+1} = 0\Big),
\end{aligned} \tag{3.3}
$$

where $Z_n := \sum_{i=1}^{n}(n-i+1)X_i$ and $Y_n := Z_n - Z_{n-1} = \sum_{i=1}^{n} X_i, n \ge 1$.

Then, central limit theorem for $(\frac{1}{N^{\frac{3}{2}}}Z_{[sN]}, \frac{1}{N^{\frac{1}{2}}}Y_{[sN]}, \frac{1}{N^{\frac{3}{2}}}Z_{[tN]}, \frac{1}{N^{\frac{1}{2}}}Z_{[tN]})$ under the measure $\mu^{(0,0)}(\,\cdot \mid Z_N = 0, Z_{N+1} = 0)$ holds (cf. [11, Theorem 2.7]) and if we denote the limiting 4-dimensional Gaussian vector as $(\zeta_s, \eta_s, \zeta_t, \eta_t)$ then the right hand side of (3.3) converges to

$$
P(\alpha \le \zeta_s \le \alpha+\beta, \alpha \le \zeta_t \le \alpha+\beta, |\eta_s| \le \beta, |\eta_t| \le \beta) = C(\alpha, \beta, s, t) > 0,
$$

as $N \to \infty$.

$\square$

Remark. [11] discussed Theorem 3.1 etc. only for one dimensional discrete height models with Laplacian interactions. But as explained in their paper, their proof also works well for continuous height models. Since the proof is essentially same, we omit it and only state the corresponding central limit theorem and its local version for our case.

Proposition 3.2. *Consider integrated random walk $Z_n := \sum_{i=1}^{n}(n-i+1)X_i$, $n \in \mathbb{N}$. Let $0 < t_1 < t_2 < \cdots < t_l \le 1$, $l \in \mathbb{N}$ and define $\eta_i := Y_{[t_iN]} = Z_{[t_iN]} - Z_{[t_iN]-1}$, $\zeta_i := Z_{[t_iN]}$ for $1 \le i \le l$.*

1. *Uniformly in $(u_1, v_1, u_2, v_2, \cdots, u_l, v_l)$ from compact sets in $\mathbb{R}^{2l}$ it holds that*

$$
\begin{aligned}
&\log E\Big[\exp\Big\{\sqrt{-1}\sum_{j=1}^{l}\big(u_j\frac{\eta_j}{\sigma\sqrt{N}} + v_j\frac{\zeta_j}{\sigma N^{\frac{3}{2}}}\big)\Big\}\Big] \\
&\quad = -\frac{1}{2}\Big\{\sum_{j=1}^{l}\sum_{k=1}^{l}(t_j \wedge t_k)u_ju_k + \sum_{j=1}^{l}\sum_{k=1}^{l}\frac{1}{6}(t_j \wedge t_k)^2\big(3(t_j \vee t_k) - (t_j \wedge t_k)\big)v_jv_k \\
&\qquad + \sum_{j=1}^{l}\sum_{k=1}^{l}\frac{1}{2}(t_j \wedge t_k)\big(2t_k - (t_j \wedge t_k)\big)u_jv_k\Big\} + o(1),
\end{aligned} \tag{3.4}
$$

as $N \to \infty$.

2. Let $f_N^{2l}(y_1, z_1, \cdots, y_l, z_l)$ *denote density of* $(\eta_1, \zeta_1, \cdots, \eta_l, \zeta_l)$, *i.e.,*

$$f_N^{2l}(y_1, z_1, \cdots, y_l, z_l) = \frac{\mu^{(0,0)}\big((\eta_1, \zeta_1, \cdots, \eta_l, \zeta_l) \in (dy_1, dz_1, \cdots, dy_l, dz_l)\big)}{dy_1 dz_1 \cdots dy_l dz_l}.$$

Then, we have

$$\lim_{N\to\infty} \sup_{(y_1,z_1,\cdots,y_l,z_l)\in\mathbb{R}^{2l}} \Big| \sigma^{2l} N^{2l} f_N^{2l}(\sigma\sqrt{N}y_1, \sigma N^{\frac{3}{2}} z_1, \cdots, \sigma\sqrt{N}y_l, \sigma N^{\frac{3}{2}} z_l) - q^{2l}(y_1, z_1, \cdots, y_l, z_l) \Big| = 0,$$

where q^{2l} *denotes density of* $2l$*-dimensional centered Gaussian vector whose covariances are determined from the right hand side of (3.4).*

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